# A New Class of Basis Functions for the Solution of the $E$-Plane Waveguide Discontinuity Problem

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*Abstract* —**A class of basis functions which satisfies the edge condition explicitly is presented to improve the convergence of the mode-matching method. Using these basis functions, we analyse the 2-to-1 $E$-plane junction, make comparisons with other available methods of solution, and provide correction terms for its quasi-static solution.**

## I. Introduction

Waveguide junction scattering problems have been studied extensively. Methods of solution include the conformal mapping [1], variational methods [2], [3], singular integral equation method [4], [5], modified residue calculus techniques [6], mode matching [7], [8] and moment methods [9], [10].

Mode matching is rigorous and systematic. But it has some shortcomings. First, it suffers from the problem of relative convergence [11]. This can be circumvented by taking a proper ratio in the truncation of infinite waveguide modes [8], [12], [13]. Another problem with mode matching is that the convergence is slow when the aperture electric field or magnetic field is singular at the edges. This been solved by Lyapin *et al.* [14] by using aperture basis functions which satisfy the edge condition explicitly. Their basis functions are, however, not very versatile. For example, they cannot be applied to the structures studied in [15] and [16]. In this paper, we present a new class of basis functions that can be applied to a fairly general type of $E$-plane discontinuity problem. In addition, we also propose correction terms for the quasi-static solutions given in [15] and [16] to improve the accuracy.

## II. Moment Method

The problem to be studied is shown in Fig. 1. The structure is quite general and it includes the cases of waveguide step (enlargement or reduction), iris, misalignment, etc. A TEM wave of unit amplitude is assumed to be incident from the left of waveguide 1. We express the total transverse electric and magnetic fields in terms of infinite waveguide modes. Matching the tangential electric and magnetic fields across the aperture and leaving the aperture electric field as the unknown, we obtain the following:

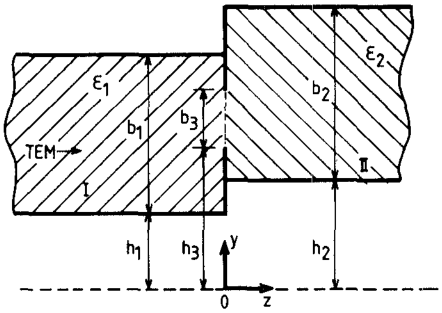


Fig. 1. $E$-plane waveguide discontinuity.

$$\sum_{n=0}^{\infty} \frac{\epsilon_n}{b_1} Y_{1n}\tilde{E}_t\left(\frac{n\pi}{b_1}, h_1\right)\cos\left[\frac{n\pi(y-h_1)}{b_1}\right] + \sum_{n=0}^{\infty}\frac{\epsilon_n}{b_2}Y_{2n}\tilde{E}_t\left(\frac{n\pi}{b_2},h_2\right)\cos\left[\frac{n\pi(y-h_2)}{b_2}\right] = 2Y_{10} \quad (1)$$

where Euler's number $\epsilon_n$, the TM mode admittances $Y_{1n}$, $Y_{2n}$, and the cosine transform of aperture field $E_t$ are defined, respectively, by the following expressions:

$$\epsilon_n = \begin{cases} 1, & n=0 \\ 2, & n\neq 0 \end{cases} \quad (2)$$

$$Y_{in} = \frac{j\epsilon_{r_i}}{120\pi\sqrt{\left(\dfrac{n}{2b_i/\lambda_0}\right)^2 - \epsilon_{r_i}}}, \qquad i=1,2 \quad (3)$$

where the complex square root is to take the value of its principal branch, and

$$\tilde{E}_t(\alpha,\beta) = \int_{h_3}^{h_3+b_3} E_t(y)\cos\left[\alpha(y-\beta)\right] dy. \quad (4)$$

We expand the aperture field in terms of the known basis function $e_m(y)$ with unknown complex magnitude $A_m$, $m=0,1,2,\cdots,M$. Applying Galerkin's method [17], we obtain a set of $(M+1)$ by $(M+1)$ algebraic equations:

$$\sum_{n=0}^{M} A_m H_{mk} = 2Y_{10}\tilde{e}_0(0,h_1), \qquad k=0,1,\cdots,M \quad (5)$$

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where

$$H_{mk} = \sum_{n=0}^{\infty} \frac{\epsilon_n}{b_1} Y_{1n} \tilde{e}_m\left(\frac{n\pi}{b_1}, h_1\right) + \sum_{n=0}^{\infty} \frac{\epsilon_n}{b_2} Y_{2n} \tilde{e}_m\left(\frac{n\pi}{b_2}, h_2\right) \tilde{e}_k\left(\frac{n\pi}{b_2}, h_2\right). \quad (6)$$

We choose the basis function to be

$$e_m(y) = k_m(1-y_1)^{-\nu}(1+y_1)^{-\mu} P_m^{(-\nu,-\mu)}(y_1) \quad (7)$$

where

$$k_m = \frac{2^{(\nu+\mu-1)}4^{-m}}{B(M+1-\nu, m+1-\mu)} \frac{2}{b_3}$$

$$y_1 = \frac{2(y-d)}{b_3}$$

$$d = h_3 + b_3/2$$

and $P_m(\alpha, \beta)$ is the Jacobi polynomial [18]. The quantities $\nu$ and $\mu$ are determined by the edge singularities at $y = b_3 + h_3$ and $y = b_3$, respectively [6], [19]. They vary from 0 (no singularity) to 0.5 (iris-type singularity). For a 90° corner, their values are given by [15] as

$$\nu \text{ or } \mu = \frac{1}{\pi} \tan^{-1}\left(\frac{\epsilon_2}{\epsilon_1}\sqrt{1+\frac{2\epsilon_1}{\epsilon_2}}\right). \quad (8)$$

The cosine transform of $e_m(y)$ as defined by (7) can be evaluated in closed form [20] as

$$\tilde{e}_m(\alpha, \beta) = \frac{4^{-m}}{m!} \operatorname{Re}\left\{\exp[-j\alpha(d-\beta)] \exp\left(j\frac{\alpha b_3}{2}\right) \cdot (-j\alpha b_3)^m \cdot {}_1F_1(m+\mu+1; 2m+\mu+\nu+2; -j\alpha b_3)\right\} \quad (9)$$

where Re{ } denotes the real part of { } and ${}_1F_1(a; b; z)$ is the confluent hypergeometric function [18, ch. 13]. A power series in $z$ and an asymptotic expansion for large $|z|$ are available in [18].

Having obtaining $A_m$'s, the reflection coefficient and the normalized junction admittance can be obtained from

$$R = \frac{A_0}{b_1} \tilde{e}_0(0,0) - 1 \quad (10)$$

$$\frac{Y}{Y_{10}} = \frac{1-R}{1+R}. \quad (11)$$

## III. 2-TO-1 $E$-PLANE JUNCTION

In this section, by way of example of the application of the basis functions, we consider the structure shown in Fig. 2, and we derive correction terms for the quasi-static formulas given in [15] and [16].

Assuming that only the dominant mode propagates, it can be shown that a variational expression for the normalized junction susceptance is given by

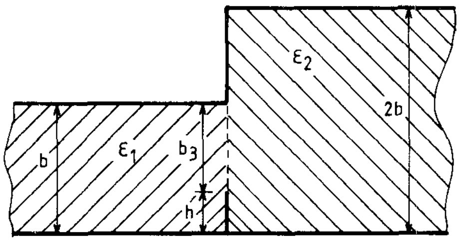


Fig. 2. 2-to-1 $E$-plane discontinuity.

$$\frac{jB}{Y_{10}} = \frac{1}{Y_{10}\tilde{E}_t^2(0,0)} \cdot \left\{\sum_{n=1}^{\infty} 2Y_{1n}\tilde{E}_t^2\left(\frac{n\pi}{b},0\right) + \sum_{n=1}^{\infty} Y_{2n}\tilde{E}_t^2\left(\frac{n\pi}{2b},0\right)\right\} \quad (12)$$

where $\tilde{E}_t$ is defined by (4) with $h_3 = h$ and $h_3 + b_3 = b$.

In the quasi-static limit, the modal admittance becomes

$$Y_{in}^{(0)} = j\frac{k_0}{\eta_0}\frac{b_i}{n\pi}\epsilon_{r_i}, \qquad i = 1,2 \quad (13)$$

where

$$k_0 = \frac{2\pi}{\lambda_0} \quad \text{and} \quad \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}.$$

Using the above, the junction susceptance can be separated into two parts: the quasi-static contribution $B_q$ and the correction term $\Delta B$ as

$$jB = jB_q + j\Delta B \quad (14)$$

where

$$jB_q = \frac{1}{\tilde{E}_t^2(0,0)}\left\{\sum_{n=1}^{\infty} 2Y_{1n}^{(0)}\left(\frac{n\pi}{b},0\right) + \sum_{n=1}^{\infty} Y_{2n}^{(0)}\tilde{E}_t^2\left(\frac{n\pi}{2b},0\right)\right\} \quad (15)$$

and

$$j\Delta B = \frac{1}{\tilde{E}_t^2(0,0)}\left\{2\left(Y_{11} - Y_{11}^{(0)}\right)\tilde{E}_t^2\left(\frac{\pi}{b},0\right) + \sum_{n=1}^{2}\left(Y_{2n} - Y_{2n}^{(0)}\right)\tilde{E}_t\left(\frac{n\pi}{2b},0\right)\right\}. \quad (16)$$

The first infinite series of $\Delta B$ has been truncated at $n = 1$ and the second series at $n = 2$.

For the case where $h = 0$ (without diaphragm), $B_q$ can be approximated by Montgomery and Lewin's quasi-static solution (ML) solution [15]:

$$\frac{B_{\text{ML}}}{Y_{10}} = \frac{2b}{\lambda_0}\sqrt{\epsilon_{r_1}}\left(\frac{2+\epsilon_2/\epsilon_1}{\epsilon_1}\right) \cdot \left[\frac{\pi}{2}\cot\beta\pi - 2\ln 2 - \gamma - \psi(1-\beta)\right] \quad (17)$$

where

$$\beta = \frac{1}{\pi}\tan^{-1}\left(1+\frac{2\epsilon_1}{\epsilon_2}\right) \quad (18)$$

$\gamma$ is the Euler constant = 0.5772, and $\psi$ is the logarithmic derivative of the gamma function.

For the correction term $\Delta B$, we approximate the aperture field by

$$E_t(y)=\sqrt{\frac{\pi}{2}}\,b\cdot\left[2^{-\nu}\Gamma(1-\nu)\left(1-\frac{y^2}{b^2}\right)^{-\nu}+\frac{\epsilon_1}{\epsilon_2}\right] \quad (19)$$

where $\nu$ is given by (8).

The first term in the square bracket gives the proper singularity at the corner. The second term is added since the aperture field is more uniform for $\epsilon_1>\epsilon_2$.

An improved ML solution can be obtained from (14), (16), and (17) with $\tilde{E}_t$ given by [20]

$$\tilde{E}_t(\alpha,0)=\frac{J_{-\nu+1/2}(b\alpha)}{(b\alpha)^{-\nu+1/2}}+\frac{\epsilon_1}{\epsilon_2}\frac{J_{1/2}(b\alpha)}{(b\alpha)^{1/2}} \quad (20)$$

where $J_\nu(x)$ is the Bessel function of the first kind of order $\nu$ [18, ch. 9].

For nonzero $d$, $B_q$ is given by Ruehle and Lewin (RL solution) [16] as

$$\frac{B_{\mathrm{RL}}}{Y_{10}}=\frac{2b}{\lambda_0}\sqrt{\epsilon_r}\left(2+\frac{\epsilon_2}{\epsilon_1}\right)\cdot\left[\frac{\pi}{2}\cot\beta\pi-2\ln 2-\gamma-\psi(1-\beta)-\ln c\right] \quad (21)$$

where

$$c=\cos\left(\frac{\pi h}{2b}\right).$$

The aperture field can be approximated by

$$E_t(y)=\sqrt{\frac{\pi}{2}}\,b_3\left\{2^{-\nu}\Gamma(1-\nu)\left(1-\frac{y^2}{b_3^2}\right)^{-\nu}+2^{-1/2}\Gamma(1/2)\frac{h}{b}\left[1-\left(\frac{b-y}{b_3}\right)^2\right]^{-1/2}+\frac{\epsilon_1}{\epsilon_2}\right\}. \quad (22)$$

The term containing $h/b$ in (22) is to account for the edge singularity at $y=h$.

The transform of $E_t$ can be found to be [20]

$$\tilde{E}_t(\alpha,0)=\frac{J_{-\nu+1/2}(b_3\alpha)}{(b_3\alpha)^{-\nu+1/2}}\cos(\alpha h)-\frac{H_{-\nu+1/2}(b_3\alpha)}{(b_3\alpha)^{-\nu+1/2}}\sin(\alpha h)+\frac{h}{b}\{J_0(b_3\alpha)\cos(\alpha b)+H_0(b_3\alpha)\sin(\alpha b)\}+\frac{\epsilon_1}{\epsilon_2}\frac{J_{1/2}(b_3\alpha)}{(b\alpha)^{1/2}} \quad (23)$$

where $H_\nu(x)$ is the Struve function [18, ch. 12]. For $h=0$, (23) reduces to (20).

TABLE I
SUSCEPTANCE OF A CAPACITIVE SEMIDIAPHRAGM AS A FUNCTION OF THE NUMBER OF BASIS FUNCTIONS ($M+1$)

| $M$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Numerical method | 1.5935 | 1.5934 | 1.5931 | 1.5931 | 1.5931 |
| Lyapin *et al.* | 1.6123 | 1.5932 | 1.5931 | 1.5931 | 1.5931 |

$h_1=h_2=h_3=0$; $b_1=b_2=2b_3$; $\lambda=2.5b_1$.

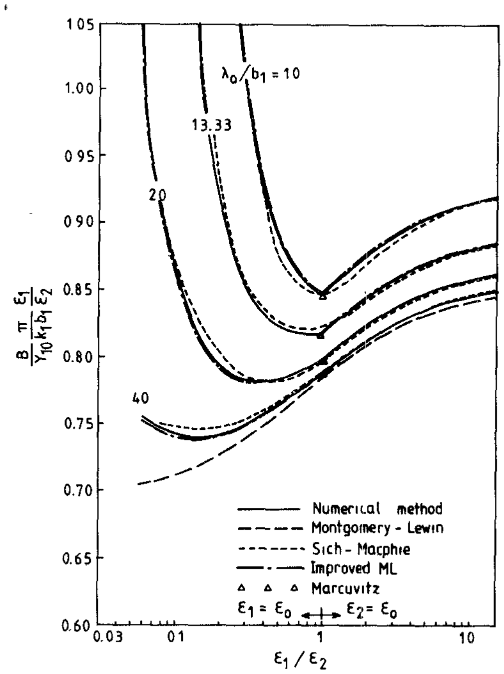


Fig. 3. Susceptance of 2-to-1 $E$-plane discontinuity: comparison of solutions.

## IV. NUMERICAL RESULTS

We have computed the susceptance of a capacitive semidiaphragm by setting $h_1=h_2=h_3=0$, $b_1=b_2=2b_3$. The results obtained from the moment method (numerical method) compare very well with the exact solution (in terms of infinite series) given in [2, p. 452]. Our convergence pattern is similar to that of Lyapin *et al.* [14]. Two to three basis functions are sufficient to produce results accurate to four or more significant figures, as shown in Table I.

We next generated results for the structure shown in Fig. 2. The junction susceptance is shown in Fig. 3 for $h=0$ (without diaphragm). Also shown are the solution by Sich and Macphie using the conservation of complex power technique [21], the quasi-static solution by Montgomery and Lewin using the singular integral equation method [15], and the conformal mapping solution given in Marcuvitz's *Waveguide Handbook* [1] (for $\epsilon_1=\epsilon_2$ only). Our numerical method agrees very well (to within 2 percent) with the improved ML solution (which is the quasi-static solution with correct terms). The agreement with the results of Sich and Macphie is fair, except near $\epsilon_1=\epsilon_2$, where our results exhibit a discontinuity of slope. We believe that the discontinuity in slope could be caused by the presence of the nonlinear higher order terms in (16) containing $Y_{11}, Y_{21}, Y_{22}$, etc. The agreement with Marcu-

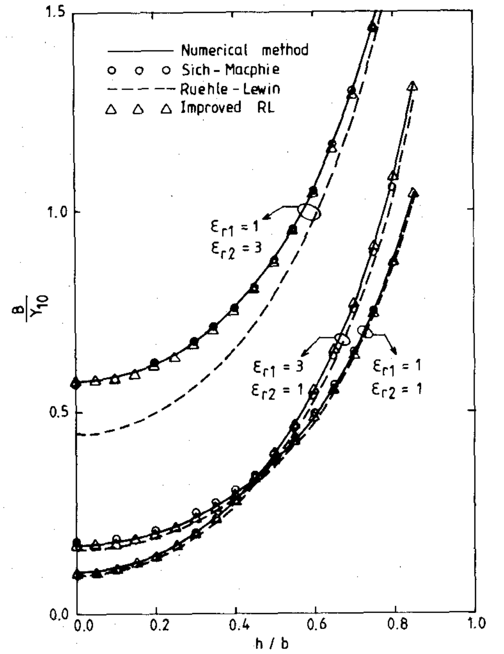


Fig. 4. Susceptance of step-diaphragm junction. $b = 0.1\,\lambda_0$.

vitz's results at $\epsilon_1 = \epsilon_2$ further confirms the accuracy of our results. Fig. 4 shows the susceptance for nonzero $h$ for three chosen sets of values of $\epsilon_{r_1}$ and $\epsilon_{r_2}$. Our results derived from the moment method agree to within 2 percent with Sich and Macphie's solution [21]. It can be seen that our improved RL solution (which is quasi-static with correction terms) also agrees very closely with the numerical method and is significantly more accurate than the quasi-static solution given by Ruehle and Lewin [16]. For $\epsilon_{r_1} = 1$ and $\epsilon_{r_2} = 3$, as shown in Fig. 4, the required correction is 22.8 percent for $h/b = 0$.

## V. Conclusions

We have proposed a new class of basis functions for the $E$-phase junction problem. They are versatile and the edge singularities can be taken into account explicitly. Numerical results have been presented for the 2-to-1 $E$-plane step-diaphragm junction. We have also proposed a method to improve the accuracy of the quasi-static solution.

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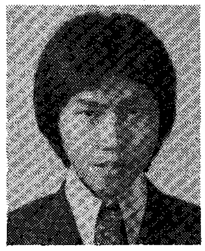

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